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A PROPOSAL OF TESTING METHOD ON THE EXACT STEADY STATE PROBABILITY DENSITY FUNCTION OF NON-LINEAR STOCHASTIC SYSTEM

R. WANG AND K. YASUDA

Department of Electronic-Mechanical Engineering, Nagoya University, Furo-Cho, Chikusa-Ku, Nagoya 464-8603, *Japan*

AND

Z. ZHANG

Shanghai Post and Telecommunication Training Centre, Shanghai, People's Republic of China

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This paper is concerned with the exact analysis method of the response process of second order non-linear stochastic systems excited by Gaussian white noise. A main feature is that a testing method is presented in this paper for the exact steady state probability density of two dimensions to demonstrate the effectiveness of the exact results. Examples are given to illustrate the applications of the analysis method.

1. INTRODUCTION

In the past 40 years the response of non-linear dynamic systems to stochastic excitations has been extensively studied, and both exact and approximate methods have been developed. When excitation is Gaussian white noise, the response of the system is a Markovian process, and the probability density of the response process is governed by the Fokker–Planck–Kolmogrov equation (the FPK equation).

In many areas of random mechanics, sometimes we need to analytically obtain the exact probability densities of response processes for non-linear stochastic systems or random oscillators. Recent results on the method of the exact FPK equation may be found in references [1–5, 6]. However, in a general case, no exact solution can be obtained and numerical methods must be used. Unfortunately, the numerical methods for solving the FPK equation in higher dimensions are very difficult to perform [7, 8], and compared with the exact analytic results, there still exist many difficulties in testing the accuracy of numerical results [5]. This comparison is especially impossible for the joint probability density made up of vector random processes of two or more dimension. Hence, this paper presents general steps to overcome the difficulties. First, the exact probability density of the response process of the non-linear stochastic system is solved in accordance with the method proposed in section 3. The main point of this method is to determine an undetermined function appearing in the reduced FPK equation by solving the Riccati equation. By using this method, an exact steady state solution of non-linear stochastic system is obtainable. Second, a non-linear stochastic differential equation is constructed in accordance with the analysis technique presented the in section 4. If a known non-linear stochastic differential equation is the same with the constructed non-linear stochastic differential equation, then the probability density corresponding to the known non-linear stochastic system is exact. Examples are given to illustrate the application of the analysis technique.

2. THE STEADY STATE FPK EQUATION

The following general non-linear system is considered:

$$\ddot{x} + g(x, \dot{x}) = w(t), \tag{1}$$

where $w(t)$ represents a zero-mean Gaussian white noise with delta-type correlation functions $E[w(t)w(t+\tau)] = 2\pi\phi\delta(\tau)$.

The stationary probability density $p(y_1, y_2)$ of the system response is governed by the reduced Fokker–Planck equation

$$-y_2\frac{\partial p}{\partial y_1} + \frac{\partial[g(y_1, y_2)p]}{\partial y_2} + \pi\phi\frac{\partial^2 p}{\partial y_2^2} = 0, \tag{2}$$

where $y_1 = x(t)$ and $y_2 = \dot{x}(t)$.

Let

$$p(y_1, y_2) = c\exp\left[-\frac{1}{\pi\phi}f(y_1, y_2)\right], \tag{3}$$

where c is a normalization constant, and $f(y_1, y_2)$ is a stationary potential function. Of course, expression (3) must be non-negative and normalizable or $p(y_1, y_2)$ to be a valid probability density

Since

$$-y_2\frac{\partial p}{\partial y_1} + \frac{\partial[(g(y_1, y_2) + g_1(y_1)/p(y_1, y_2))p(y_1, y_2)]}{\partial y_2} + \pi\phi\frac{\partial^2 p}{\partial y_2^2}$$

$$= -y_2\frac{\partial p}{\partial y_1} + \frac{\partial[g(y_1, y_2)p(y_1, y_2)]}{\partial y_2} + \pi\phi\frac{\partial^2 p}{\partial y_2^2}, \tag{4}$$

$g(y_1, y_2)$ and $g(y_1, y_2) + g_1(y_1)/p(y_1, y_2)$ have the same probability density. Substituting equation (3) into equation (2) yields

$$p\left(y_2\frac{f_{y_1}}{\pi\phi} - g\frac{f_{y_2}}{\pi\phi} + \frac{f_{y_2}^2}{\pi\phi} + g_{y_2} - f_{y_2y_2}\right) = 0, \tag{5}$$

where

$$f_{y_1} = \frac{\partial f}{\partial y_1}, \qquad f_{y_2} = \frac{\partial f}{\partial y_2},$$

$$f_{y_2}^2 = \left(\frac{\partial f}{\partial y_2}\right)^2, \qquad f_{y_2 y_2} = \frac{\partial^2 f}{\partial y_2^2}, \qquad g_{y_2} = \frac{\partial g}{\partial y_2}$$

Since $p(y_1, y_2) \neq 0$, equation (5) is reduced to

$$\frac{\partial g}{\partial y_2} - \frac{f_{y_2} g}{\pi\phi} + y_2 \frac{f_{y_1}}{\pi\phi} + \frac{f_{y_2}^2}{\pi\phi} - f_{y_2 y_2} = 0. \tag{6}$$

3. THE EXACT ANALYSIS METHOD OF THE FPK EQUATION

If $f(y_1, y_2)$ satisfies the following equation set:

$$f_{y_2 y_2} - g_{y_2} = \frac{1}{\pi\phi} h(y_1, y_2), \tag{7}$$

$$y_2 f_{y_1} - g f_{y_2} + f_{y_2}^2 = h(y_1, y_2), \tag{8}$$

then $f(y_1, y_2)$ can satisfy the FPK equation (6) where $h(y_1, y_2)$ is an undetermined function. The above equations show balance of the probability potential function.

We can obtain from equation (7),

$$f_{y_2} = g(y_1, y_2) - g(y_1, 0) + f_{y_2}(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2, \tag{9}$$

$$f = \int_0^{y_2} g(y_1, y_2)\,\mathrm{d}y_2 - y_2 g(y_1, 0) + y_2 f_{y_2}(y_1, 0) + f(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2, \tag{10}$$

$$f_{y_1} = \int_0^{y_2} g_{y_1}\,\mathrm{d}y_2 - y_2 g_{y_1}(y_1, 0) + y_2 f_{y_2 y_1}(y_1, 0) + f_{y_1}(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2}\int_0^{y_2} h_{y_1}(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2. \tag{11}$$

Substituting equations (9) and (11) into equation (8) yields

$$y_2\left[\int_0^{y_2} g_{y_1}\,\mathrm{d}y_2 - y_2 g_{y_1}(y_1, 0) + y_2 f_{y_2 y_1}(y_1, 0) + f_{y_1}(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2}\int_0^{y_2} h_{y_1}(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2\right] - \left[g(y_1, 0) - f_{y_2}(y_1, 0) - \frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2\right]\left[g(y_1, y_2) - g(y_1, 0) + f_{y_2}(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2\right] = h(y_1, y_2). \tag{12}$$

Assuming $f_{y_2}(y_1, 0) = 0$, equation (12) may be expressed as

$$f_{y_1}(y_1, 0) = \frac{g(y_1, 0)}{y_2}[g(y_1, y_2) - g(y_1, 0)] + y_2 g_{y_1}(y_1, 0) - \int_0^{y_2} g_{y_1}(y_1, y_2)\,\mathrm{d}y_2 + h_1(y_1, y_2), \tag{13}$$

where

$$h_1(y_1, y_2) = \frac{h(y_1, y_2)}{y_2} - \frac{1}{\pi\phi}\int_0^{y_2}\int_0^{y_2} h_{y_1}(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2 - \frac{1}{\pi\phi y_2}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2 \times\left[g(y_1, y_2) - 2g(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2\right]. \tag{14}$$

The function $h(y_1, y_2)$ is selected in order to make the right-hand side of equation (13) to be a function of y_1. Hence, equation (13) may be further expressed as

$$f_{y_1}(y_1, 0) = k_1(y_1) + k_2(y_1, y_2) + h_1(y_1, y_2) \tag{15}$$

where

$$k_1(y_1) + k_2(y_1, y_2) = \frac{g(y_1, 0)}{y_2}[g(y_1, y_2) - g(y_1, 0)] + y_2 g_{y_1}(y_1, 0) - \int_0^{y_2} g_{y_1}(y_1, y_2)\,\mathrm{d}y_2.$$

Let

$$k_2(y_1, y_2) + h_1(y_1, y_2) = h_2(y_1), \tag{16}$$

The above results are substituted into equation (15) yielding

$$f_{y_1}(y_1, 0) = k_1(y_1) + h_2(y_1). \tag{17}$$

Substituting equation (17) into equation (10) and combining equation (14) yield

$$f(y_1, y_2) = \int_0^{y_2} g(y_1, y_2)\,\mathrm{d}y_2 - y_2 g(y_1, 0) + \int_0^{y_1} [k_1(y_1) + h_2(y_1)]\,\mathrm{d}y_1 + \frac{1}{\pi\phi}\Gamma(y_1, y_2), \tag{18}$$

where $h_2(y_1)$ satisfies (the combination in equation (14))

$$\frac{\Gamma_{y_2 y_2}}{y_2} - \frac{\Gamma_{y_1}}{\pi\phi} - \frac{\Gamma_{y_2}}{\pi\phi y_2}\left[g - 2g(y_1, 0) + \frac{\Gamma_{y_2}}{\pi\phi}\right] = h_2(y_1) - k_2(y_1, y_2), \tag{19}$$

with

$$\Gamma(y_1, y_2) = \int_0^{y_2}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2.$$

Solving $h_2(y_1)$ is generally difficult from equation (19). An alternative method is given below.

By the assumption of the equation set

$$\frac{\Gamma_{y_2 y_2}}{y_2} = h_2(y_2), \tag{20}$$

$$\frac{\Gamma_{y_1}}{\pi\phi} + \frac{\Gamma_{y_2}}{\pi\phi y_2}\left[g - 2g(y_1, 0) + \frac{\Gamma_{y_2}}{\pi\phi}\right] = k_2(y_1, y_2), \tag{21}$$

the solution of equations (20) and (21) satisfy equation (19). Integrating equation (20) yields

$$\Gamma_{y_2} = \frac{y_2^2}{2} h_2(y_1), \qquad \Gamma = \frac{y_2^3}{6} h_2(y_1), \qquad \Gamma_{y_1} = \frac{y_2^3}{6} h_2'(y_1), \tag{22–24}$$

Substituting equations (22) and (24) into equation (21), we obtain the following Riccati equation:

$$h_2'(y_1) + \frac{3}{2\pi\phi} h_2^2(y_1) + \frac{3}{y_2^2}[g(y_1, y_2) - 2g(y_1, 0)]h_2(y_1) = \frac{6\pi\phi}{y_2^3} k_2(y_1, y_2). \tag{25}$$

In general cases, equation (25) cannot be used to express elementary functions. But, if $h_2(y_1)$ is considered to be a function of y_1, then the remaining terms of the above equation, namely,

$$\frac{3}{y_2^2}[g(y_1, y_2) - 2g(y_1, 0)]h_2(y_1) - \frac{6\pi\phi}{y_2^3} k_2(y_1, y_2) \tag{26}$$

must be a function of y_1. If function $h_2(y_1)$ exists in formula (26), we can solve function $h_2(y_1)$ from (25). The result is then substituted into equation (25).

If $h_2(y_1)$ cannot be found, or substituting $h_2(y_1)$ into equation (25) will not be established, then it is shown that the conditions assumed in equations (20) and (21) are not valid. In this case, other conditions must be assumed.

If equation (23) and $h_2(y_1)$ satisfy the Riccati equation (25), the results of $h_2(y_1)$ are substituted into equation (18) and yield the following exact result.

$$f(y_1, y_2) = \int_0^{y_2} g(y_1, y_2)\,\mathrm{d}y_2 - y_2 g(y_1, 0) + \int_0^{y_1} [k_1(y_1) + h_2(y_1)]\,\mathrm{d}y_1 + \frac{y_2^3}{6\pi\phi} h_2(y_1). \tag{27}$$

Example 1. Consider the following non-linear stochastic system:

$$\ddot{x} + 2b\frac{\dot{x}}{x^{2n-2}} - (n-1)\frac{\dot{x}^2}{x} + \frac{an}{b}x^{4n-3} - \frac{\pi\phi(n-1)}{b}x^{2n-3} = w(t). \tag{28}$$

Then $g(y_1, y_2) = 2b(y_2/y_1^{2n-2}) - (n-1)y_2^2/y_1 + ((an/b)y_1^{4n-3}) - (\pi\phi(n-1)/b)$ y_1^{2n-3}, where a and b are two positive constants.

Substituting the above equations into equation (15), we get

$$k_1(y_1) + k_2(y_1, y_2) = 2\left[any_1^{2n-1} - \frac{\pi\phi(n-1)}{y_1}\right] - \frac{(n-1)}{b}$$
$$\times y_2[any_1^{4n-4} - \pi\phi(n-1)y_1^{2n-4}]$$
$$+ 2(n-1)b\frac{y_2^2}{y_1^{2n-1}} - (n-1)\frac{y_2^3}{3y_1^2},$$

$$\text{thus } k_1(y_1) = 2\left[any_1^{2n-1} - \pi\phi\frac{n-1}{y_1}\right] \tag{29}$$

and

$$k_2(y_1, y_2) = -\frac{(n-1)}{b}y_2[any_1^{4n-4} - \pi\phi(n-1)y_1^{2n-4}]$$
$$+ 2(n-1)b\frac{y_2^2}{y_1^{2n-1}} - (n-1)\frac{y_2^3}{3y_1^2}. \tag{30}$$

Furthermore, it is desirable for $h_2(y_1)$ to be derived from equation (26):

$$\frac{3}{y_2^2}\left\{\left[2b\frac{y_2}{y_1^{2n-2}} - (n-1)\frac{y_2^2}{y_1} - \frac{an}{b}y_1^{4n-3} + \frac{\pi\phi(n-1)}{b}y_1^{2n-3}\right]h_2(y_1)\right.$$
$$+ 2\pi\phi(n-1)\frac{y_2^2}{3y_1^2} - 4\pi\phi(n-1)\,b\frac{y_2}{y_1^{2n-1}}$$
$$\left.+\frac{2\pi\phi(n-1)}{b}[any_1^{4n-4} - \pi\phi(n-1)y_1^{2n-4}]\right\} = -\frac{3(n-1)}{y_1}\left[h_2(y_1) - \frac{2\pi\phi}{3y_1}\right]$$
$$+ \frac{3}{y_2^2}\left[h_2(y_1) - 2\pi\phi\frac{(n-1)}{y_1}\right]$$
$$\times\left[\frac{\pi\phi(n-1)}{b}y_1^{2n-3} - \frac{an}{b}y_1^{4n-3} + 2b\frac{y_2}{y_1^{2n-2}}\right].$$

When

$$h_2(y_2) = 2\pi\phi\,\frac{(n-1)}{y_1}, \tag{31}$$

equation (26) becomes $-[6\pi\phi/y_1^2](n-1)(n-4/3)$ to be only the function of y_1. The results obtained by substituting equations (30) and (31) into equation (25) satisfy the Riccati equation (25).

Substituting equations (29) and (31) into equation (27), we get

$$f(y_1, y_2) = ay_1^{2n} + b\,\frac{y_2^2}{y_1^{2n-2}}. \tag{32}$$

By making use of equation (3), the exact steady state probability density of the non-linear system defined by equation (28) is given by

$$p(y_1, y_2) = c\exp\left[-\frac{1}{\pi\phi}\left(ax^{2n} + b\frac{\dot{x}^2}{x^{2n-2}}\right)\right], \tag{33}$$

where the normalization constant c can be solved as follows:

$$c = \frac{\sqrt{abn}}{\pi^2\phi}. \tag{34}$$

When $n = 1$, equation (28) becomes the following linear form:

$$\ddot{x} + 2b\dot{x} + \frac{a}{b}x = w(t). \tag{35}$$

The exact probability density of the above equation is

$$p(x, \dot{x}) = c\exp\left[-\frac{1}{\pi\phi}(ax^2 + b\dot{x}^2)\right], \tag{36}$$

the result is well-known.

It should be pointed out that Zhu [9] has obtained an equation of motion of the exact stationary solutions of the relative general single-degree-of-freedom (s.d.o.f) non-linear stochastic systems on the basis of results studied by Caughey and Ma [10] as well as Young and Lin [11]. Here, we first show that equation (28) does not belong to the stochastic differential equation of this type presented by Zhu [9]. Next, we show that some concrete non-linear stochastic systems are found to be still very difficult by means of this equation of motion presented by Zhu [9]. The extent of the difficulties is no less than trying to find a solution for the FPK equation [4]. The method provided by the paper can overcome these difficulties.

Example 2. Consider the following non-linear system:

$$\ddot{x} + g_0(x) + (1 + x^2)\dot{x} + \frac{x\dot{x}^2}{1 + x^2} = w(t). \tag{37}$$

Then $g(y_1, y_2) = g_0(y_1) + (1 + y_1^2)y_2 + y_1 y_2^2/(1 + y_1^2)$.

By combining the above equation and equation (15), we get

$$k_1(y_1) + k_2(y_1, y_2) = g_0(1 + y_1^2) + g_0\frac{y_1 y_2}{1 + y_1^2} - y_1 y_2^2 - \frac{y_2^3}{3(1 + y_1^2)} + \frac{2y_1^2 y_2^3}{3(1 + y_1^2)^2}.$$

Thus

$$k_1(y_1) = g_0(1 + y_1^2), \tag{38}$$

$$k_2(y_1, y_2) = g_0\frac{y_1 y_2}{1 + y_1^2} - y_1 y_2^2 - \frac{y_2^3}{3(1 + y_1^2)} + \frac{2y_1^2 y_2^3}{3(1 + y_1^2)^2}. \tag{39}$$

By substituting equation (39) into equation (26), one can find $h_2(y_1)$,

$$\frac{3}{y_2^3}\left[h_2(y_1)+\frac{2\pi\phi y_1}{1+y_1^2}\right][(1+y_1^2)y_2-g_0(y_1)]+3y_1\frac{h_2(y_1)}{1+y_1^2}+\frac{2\pi\phi}{1+y_1^2}-\frac{4\pi\phi y_1^2}{(1+y_1^2)^2}.$$

When

$$h_2(y_1)=-\,2\pi\phi\frac{y_1}{1+y_1^2}, \tag{40}$$

equation (26) becomes $2\pi\phi/(1+y_1^2)-10\pi\phi\, y_1^2/(1+y_1^2)^2$ to be a function of y_1.

The results obtained by substituting equations (38) and (40) into equation (25) satisfy the Riccati equation (25).

Substituting equations (38) and (40) into equation (27) yields

$$f(y_1,y_2)=-\,\pi\phi\ln(1+y_1^2)+\frac{1+y_1^2}{2}y_2^2+\int_0^{y_1}g_0(y_1)(1+y_1^2)\,\mathrm{d}y_1. \tag{41}$$

When $g_0(x)$ in equation (37) possesses certain form, this equation may not belong to the equation of motion of the exact stationary solutions of the relative general s.d.o.f. non-linear stochastic system presented by Zhu [9].

Example 3. Consider the following non-linear oscillator [10]:

$$\ddot{x}+\frac{\beta\dot{x}}{1+\dot{x}^2/2}+\frac{1+\dot{x}^2/2}{1+x^2/2}=w(t). \tag{42}$$

Then

$$g(y_1,y_2)=\frac{\beta y_2}{1+y_2^2/2}+\frac{1+y_2^2/2}{1+y_1^2/2}$$

By combining the above equation and equation (15), we get

$$k_1(y_1)+k_2(y_1,y_2)=\frac{\beta y_1}{1+y_1^2/2}-\frac{\beta y_1 y_2^2}{(2+y_1^2)(1+y_2^2/2)}+\frac{y_1^2y_2/2-(y_2^3/6)(1-y_1^2/2)}{(1+y_1^2/2)^2}.$$

Then

$$k_1(y_1)=\frac{\beta y_1}{1+y_1^2/2}, \tag{43}$$

$$k_2(y_1,y_2)=-\,\beta\frac{y_1y_2^2}{(2+y_1^2)(1+y_2^2/2)}+\left[\frac{y_1^2y_2/2-y_2^3/6\,(1-y_1^2/2)}{(1+y_1^2/2)^2}\right]. \tag{44}$$

Under the above condition, equation (26) becomes

$$\frac{3y_1h_2(y_1)}{1+y_1^2/2}+\pi\phi\cdot\frac{1-y_1^2/2}{(1+y_1^2/2)}+\frac{3}{y_2^2}\left[h_2(y_1)+\pi\phi\frac{y_1}{1+y_1^2/2}\right]\left[\frac{\beta y_2}{1+y_2^2/2}-\frac{y_1}{1+y_1^2/2}\right]. \tag{45}$$

When

$$h_2(y_1) = -\pi\phi \frac{y_1}{1 + y_1^2/2}, \tag{46}$$

equation (45) becomes $\pi\phi\ (1 - 2y_1^2)/(1 + y_1^2/2)^2$ to be a function of y_1.

The results obtained by substituting equations (43) and (46) into equation (25) satisfy the Riccati equation.

By making use of equations (27), (43) and (46), we get

$$f(y_1, y_2) = \ln\left[\left(1 + \frac{y_1^2}{2}\right)^{\beta - \pi\phi}\left(1 + \frac{y_2^2}{2}\right)^{\beta}\right], \tag{47}$$

where $\beta > 2\pi\phi$.

The above result is in complete accordance with the solution obtained by Caughey and Ma [9].

4. A TESTING TECHNIQUE OF THE EXACT STEADY STATE PROBABILITY DENSITY

We shall now turn our attention to the testing method of the above exact steady state probability density, so that we recall equation (6) in section 2 as follows:

$$\frac{\partial g}{\partial y_2} - \frac{f_{y_2} g}{\pi\phi} + y_2 \frac{f_{y_1}}{\pi\phi} + \frac{f_{y_2}^2}{\pi\phi} - f_{y_2 y_2} = 0. \tag{48}$$

Since $g(y_1, y_2)$ is a function of $f(y_1, y_2)$ in equation (48), equation (48) is a first order linear differential equation with respect to $g(y_1, y_2)$. Then an exact general solution of equation (48) is

$$g(y_1, y_2) = f_{y_2}(y_1, y_2) - \frac{1}{\pi\phi}\exp\left[\frac{1}{\pi\phi} f(y_1, y_2)\right]\int_0^{y_2} y_2 f_{y_1}(y_1, y_2) \times \exp\left[-\frac{1}{\pi\phi} f(y_1, y_2)\right]\mathrm{d}y_2. \tag{49}$$

Example 1. The exact result of equation (28), which is given in Equation (32) as

$$f(y_1, y_2) = a y_1^{2n} + b\frac{y_2^2}{y_1^{2n-2}}, \tag{50}$$

where a and b are two positive constants, will be treated here.

By substituting the above equation into equation (49), we get

$$g(y_1, y_2) = 2b\frac{y_2}{y_1^{2n-2}} - (n-1)\frac{y_2^2}{y_1} + \frac{an}{b} y_1^{4n-3} - \frac{\pi\phi(n-1)}{b} y_1^{2n-3}. \tag{51}$$

Namely, the non-linear stochastic system is

$$\ddot{x} + 2b\frac{\dot{x}}{x^{2n-2}} - (n-1)\frac{\dot{x}^2}{x} + \frac{an}{b}x^{4n-3} - \frac{\pi\phi(n-1)}{b}x^{2n-3} = w(t). \tag{52}$$

It is shown that equation (52) is in complete accordance with equation (28) in section 3. Hence, the steady state probability density defined by equation (28) is exact.

Example 2. The following exact results obtained by Caughey and Ma [10] are tested, here the exact results are given by

$$f(y_1, y_2) = \pi\phi\left(\int_0^H h(u)\,\mathrm{d}u - \ln H_Y\right), \tag{53}$$

where $Y = y_2^2/2$ presents the kinetic energy of the system, and $H = H(y_1, y_2)$ presents the total energy of the system. It is a first integration of the equation $\ddot{y}_1 + H_{y_1}/H_Y = 0$.

By combining equations (49) and (53) yields

$$\begin{aligned} g(y_1, y_2) &= \pi\phi\left[h(H)H_Y - \frac{H_{YY}}{H_Y}\right]y_2 - \frac{1}{H_Y}\exp\left[\int_0^H h(u)\,\mathrm{d}u\right] \\ &\quad\times\int y_2\left[h(H)H_{y_1} - \frac{H_{Yy_1}}{H_Y}\right]\exp\left[-\int_0^H h(u)\,\mathrm{d}u\right]H_Y\,\mathrm{d}y_2 \\ &= \pi\phi\left[h(H)H_Y - \frac{H_{YY}}{H_Y}\right]y_2 - \frac{\exp\left[\int_0^H h(u)\,\mathrm{d}u\right]}{H_Y}\int[H_Y h(H)H_{y_1} - H_{Yy_1}] \\ &\quad\times\exp\left[-\int_0^H h(u)\,\mathrm{d}u\right]\mathrm{d}Y \end{aligned} \tag{54}$$

because

$$\begin{aligned} \int H_{Yy_1}\exp\left[-\int_0^H h(u)\,\mathrm{d}u\right]\mathrm{d}Y &= H_{y_1}\exp\left[-\int_0^H h(u)\,\mathrm{d}u\right] + \int H_y h(H)H_{y_1} \\ &\quad\times\exp\left[-\int_0^H h(u)\,\mathrm{d}u\right]\mathrm{d}Y \end{aligned} \tag{55}$$

Substituting equation (55) into equation (54) yields

$$g(y_1, y_2) = \pi\phi y_2\left[h(H)H_Y - \frac{H_{YY}}{H_Y}\right] + \frac{H_{Y_1}}{H_Y}. \tag{56}$$

The resulting non-linear system now becomes

$$\ddot{x} + \pi\phi\left[h(H)H_Y - \frac{H_{YY}}{H_Y}\right]\dot{x} + \frac{H_x}{H_Y} = w(t). \tag{57}$$

It is shown that equation (57) is in complete accordance with equation (4) by Caughey and Ma [10]. Hence, the result obtained by reference [10] is exact, namely, the exact steady state probabilistic density of the above equation is shown in the resulting equation

$$p(x, \dot{x}) = AH_Y \exp\left[-\int_0^H h(u)\,\mathrm{d}u\right]. \tag{58}$$

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5. CONCLUDING REMARKS

This paper is aimed at developing an analysis method of non-linear random mechanics. In a general case, exact solutions for randomly excited non-linear systems are difficult to obtain mathematically, and there has been no uniform analysis method for the response of a non-linear stochastic system. For this reason, different exact analysis methods have been developed for different non-linear stochastic systems. In this paper the main points are the following:

1. The most important aspect of this paper is to present a testing method for the exact steady state probability density of two dimensions in order to demonstrate the effectiveness of the exact results. Examples are given to show that this testing method is effective. Moreover, this testing method can be generalized to the non-linear stochastic system of higher dimensions.
2. By the proposed method, non-linear damping of various types can be treated.
3. This paper presents the exact analysis method, which when applied to a system treated by Caughey and Ma, yields the same solution obtained by them.
4. If an undetermined function in a non-linear stochastic system can satisfy the Riccati equation (25), then exact steady state solutions are obtainable.
5. When some concrete non-linear stochastic systems cannot satisfy the relative general motion equation of the s.d.o.f. non-linear stochastic system presented by Zhu [9], obtaining an exact solution is still possible. Of course Zhu's method can also solve the equation that the method of this paper cannot solve.

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